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The superfield formalism applied to the non-commutative Wess-Zumino model

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Abstract: We introduce the notion of superoperators on non-commutative $\mathbb{R}^4$ and re-investigate in the framework of superfields the non-commutative Wess-Zumino model as a quantum field theory. In a highly efficient manner we are able to confirm the result that this model is renormalizable to all orders.

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Contents

1. Introduction

There are no doubts that the concept of space-time as a differentiable manifold cannot be extrapolated to extremely short distances [1]. Simple heuristic arguments combining the principles of both general relativity and quantum theory imply that it is impossible to locate a particle with an arbitrarily small uncertainty [2]. This means that standard differential geometry is certainly not an adapted method for physics at short distances. On the other hand, our standard description of fundamental interactions is exclusively based upon standard differential geometry — fibre bundles for the standard model and Riemannian geometry for gravity.

If standard differential geometry is not appropriate — what else should replace it? A promising candidate is *non-commutative geometry* pioneered by Alain Connes [3], see [4] for a recent review. An excellent book [5] on this subject will soon appear. Non-commutative geometry is the attempt to extend the principles of quantum mechanics to geometry itself: the use of operator algebras, Hilbert spaces, functional analysis. Within this framework the analogue of gauge theory has been developed which reduces to Yang-Mills theory if the geometry is commutative. The first example of such a theory on a non-commutative space appeared almost ten years ago, when Connes and Rieffel developed classical two dimensional Yang-Mills theory on the non-commutative torus [6]. There is also an interesting class of "almost commutative" geometries which allow to treat Yang-Mills and Higgs fields on an equal footing and lead to a new understanding of spontaneous symmetry breaking, see e.g. [7]. The analogue of external field quantization on non-commutative spaces was proposed in [8].

Thus, the strategy of non-commutative geometry is to generalize the mathematical structures encountered in experimentally confirmed physics. Another approach to short-distance physics is string theory/ M-theory which tries to guess physics from new first principles. At first sight it seems unlikely that non-commutative geometry and string theory could be related. However, it has been shown that certain non-commutative geometries arise as limiting cases of string theory. The first hint came from [9] where compactifications of M-theory on the non-commutative torus were introduced, leading to the interpretation that the step from the commutative to the non-commutative torus corresponds to turning on a constant background 3-form C. Then, in [10] it was shown that this situation is obtained by starting with a type-IIa superstring theory with non-zero Neveu-Schwarz B-field and taking a scaling limit according to [11, 12]. That idea was thoroughly investigated in [13]. Using the results of [14] about instantons on non-commutative $\mathbb{R}^4$, Seiberg and Witten argued that there is an equivalence between the Yang-Mills theories on standard $\mathbb{R}^4$ and $\mathbb{R}^4_{nc}$.

It should be mentioned that matrix theories were studied long before M-theory was proposed, and that these matrix theories did contain certain non-commutative features. These models live on a lattice, and the number of degrees of freedom is reduced when the size N of the matrix goes to infinity [15]. Putting them on a torus instead of a lattice, twisted boundary conditions [16] are possible. Then the action can be rewritten in terms of non-commuting matrix derivatives $[\Gamma^{(j)}, \,.\,]$, with $[\Gamma^{(2j)}, \Gamma^{(2j+1)}] = -2\pi \mathrm{i}/N$, see [17].

The Seiberg-Witten paper [13] inspired numerous attempts to formulate quantum field theories on non-commutative geometries. Nevertheless, quantum field theory on non-commutative spaces is also interesting in its own right. As standard quantum field theory is the art to deal with problems of interactions at short distances, see e.g. the proceedings [18], one should expect interesting features when doing quantum field theory on spaces with different short-distance structure. Singularities in standard quantum field theories are a consequence of the point-like interactions. There has been some hope that smearing out the points it is possible to avoid these ultraviolet divergences. Example of geometries where points are replaced by some sort of cells are the fuzzy spaces, see e.g. [19, 20]. Such fuzzy spaces also arise as limits of brane dynamics [21].

That divergences are not avoided on $\mathbb{R}^4_{nc}$ was first noticed by Filk [22]. He showed that the non-commutative model contains Feynman graphs which are identical with their commutative counterparts. The $\mathbb{R}^4_{nc}$ is defined by the following commutator of the coordinate operators $\{q^\mu\}$:

$$[q^\mu, q^\nu] = \mathrm{i}\Sigma^{\mu\nu}\,, \qquad \Sigma^{\mu\nu} = -\Sigma^{\nu\mu} \in \mathbb{R}\,.$$

Integrals corresponding to Feynman graphs in non-commutative QFTs differ from their commutative counterparts by phase factors $\mathrm{e}^{\frac{\mathrm{i}}{2}\Sigma^{\mu\nu}p_\mu k_\nu}$, where p, k are internal or external momenta. The case $p = k$ is possible, and in this situation the integrals of the commutative and the non-commutative theory coincide.

This raised the question whether the non-commutative QFT is renormalizable. On the one-loop level this was affirmed for Yang-Mills theory on $\mathbb{R}^4_{nc}$ [23] and the non-commutative 4-torus [24] as well as for supersymmetric Yang-Mills theory in $(2+1)$ dimensions, with space being the non-commutative 2-torus [25]. Quantum electrodynamics on $\mathbb{R}^4_{nc}$ was treated in [26] and the BF-Yang-Mills theory in [27].

These results lead to the hope that Yang-Mills theory on $\mathbb{R}^4_{nc}$ is renormalizable to all orders in perturbation theory. It was shown however by Minwalla, Van Raamsdonk and Seiberg [28] that at least for scalar theories (ϕ^4 on $\mathbb{R}^4_{nc}$ and ϕ^3 on $\mathbb{R}^6_{nc}$) there is a new type of infrared divergences which ruins the perturbative renormalization beyond two loop. This follows immediately from the work of Filk [22], it was nevertheless completely unexpected: the oscillatory factors $\mathrm{e}^{\frac{\mathrm{i}}{2}\Sigma^{\mu\nu}p_\mu k_\nu}$ render in four dimensions an otherwise (superficially) divergent integral convergent if both p, k are internal momenta. If e.g. p is external and k is internal, the integral is convergent as well, but of course only as long as $p \neq 0$, where the ultraviolet divergence of the commutative theory reappears. This manifests as an infrared divergence coming from a ultraviolet-dangerous integration (UV/IR mixing). It turns out that the power counting degree of the new superficial IR divergence of the non-commutative theory coincides with the degree of the superficial UV divergence of the commutative theory [29].

Inspite of some rumour in the literature that Yang-Mills theory has only logarithmic divergences, it turned out that Yang-Mills on $\mathbb{R}^4_{nc}$ has quadratic IR divergences for the gluon propagator and linear IR divergences for the 3-gluon vertex [31] which prevent the perturbative renormalization. This result was also derived from the scaling limit of string theory [32, 33]. Graphs made exclusively of nested ghost propagator corrections have only logarithmic divergences and are renormalizable at any loop order [30]. In [31] it was also shown that adding fermions in the adjoint representation cancels the quadratic and linear IR divergences at one loop. This was a hint that supersymmetric Yang Mills could be renormalizable, which is also strongly supported by the divergence analysis in [29].

The UV/IR mixing was also observed in non-commutative complex scalar ϕ^4 theory [34] where the interaction potential $a\phi^*\phi\phi^*\phi + b\phi^*\phi^*\phi\phi$ is only one-loop renormalizable for $b = 0$ or $a = b$. UV/IR mixing does also occur in non-relativistic models [35]. Anomalies were investigated in [36] and the operator product expansion in [37]. Discrete symmetries (CPT) were investigated in [38].

A completely finite model is Chern-Simons theory on $\mathbb{R}^3_{nc}$ [39]. Examples of models where the limit $\Sigma^{\mu\nu} \to 0$ is smooth can be found in [40]. Yang-Mills with fermions on $\mathbb{R}^3_{nc}$ was studied in [41] with respect to the induced Chern-Simons action. Non-commutative topological massive Yang-Mills was treated in [42]. There are also interesting two-dimensional non-commutative models such as the non-linear σ model [43] and the non-commutative Wess-Zumino-Witten model [44].

Even at the tree-level a field theory on $\mathbb{R}^4_{nc}$ shows unusual features such as violation of causality [45] and S-matrix unitarity [46] if time is non commutative. Experimental limits on $\Sigma^{\mu\nu}$ are discussed in [47]. Beside the renormalizability problem, a quadratic IR divergence in the one-loop gluon propagator leads to a confinement of the model to a region whose size is of the order $\sqrt{|\Sigma^{\mu\nu}|}$. Thus, such a model cannot have the standard model as its low energy limit. The standard model can only be extended to non-commutative space-time if the quadratic IR divergences cancel, which is probably the case of supersymmetric versions. This motivates the interest in supersymmetry on non-commutative spaces, apart from the scaling limit of superstring theory.

Concerning supersymmetry, in [48] a deformation also of the anticommutator of the fermionic superspace coordinates θ was considered, it was shown however that such a deformation is not compatible with supertranslations and chiral superfields. This result was also derived in [49] via the scaling limit of string theory. A superspace formulation (at the classical level) of the Wess-Zumino model and super Yang-Mills was given in [50]. Employing the component formulation it was eventually proved in [51] that the Wess-Zumino (WZ) model on $\mathbb{R}^4_{nc}$ is renormalizable to all orders in perturbation theory. One-loop renormalizability of $N = 2$ super Yang-Mills was obtained in [52].

In this paper we extend the work of Filk [22] to the superfield formalism and apply the techniques to the non-commutative Wess-Zumino model. The paper is organized as follows: in section 2, following closer the work of Filk than [48, 49, 50], we introduce the notion of a superoperator on non-commutative space-time. In section 3 the Wess-Zumino model in the superfield formalism is introduced at the classical level, and in section 4 we postulate a non-commutative version of the Gell-Mann Low formula, and apply it to compute the one loop self-energy contributions. To be complete, we deduce the superfield Feynman rules in section 5, and apply them to the one loop vertex corrections. In section 6 we discuss the renormalizability to all loop orders.

2. Superfields with non-commutative coordinates

Following [22, 2] we consider the space-time coordinates of a flat space as self-adjoint operators in a Hilbert space with the following algebra:

$$[q^\mu, q^\nu] = \mathrm{i}\Sigma^{\mu\nu}\,, \qquad [\Sigma^{\mu\nu}, q^\rho] = 0\,, \tag{2.1}$$

where $\Sigma^{\mu\nu}$ is real and antisymmetric. In order to describe the superfields consistently within the framework of Filk one defines the operator

$$\mathrm{T}(k) = e^{\mathrm{i}k_\mu q^\mu}\,, \tag{2.2}$$

with the properties

$$T^+(k) = \mathrm{T}(-k)\,, \tag{2.3}$$

$$\mathrm{T}(k)\,\mathrm{T}(k') = T(k+k')e^{-\mathrm{i}k\times k'}\,, \qquad k\times k' = \frac{1}{2}\Sigma^{\mu\nu}k_\mu k'_\nu\,, \tag{2.4}$$

and formally

$$\mathrm{tr}\,[\mathrm{T}(k)] = \int d^4q\, e^{\mathrm{i}k_\mu q^\mu} = \prod_\mu \delta(k_\mu)(2\pi)^4\,. \tag{2.5}$$

Additionally, we are dealing with classical chiral and anti-chiral superfields $\Phi(x,\theta,\bar\theta)$ and $\bar\Phi(x,\theta,\bar\theta)$ in the real representation [53], defined on an ordinary manifold by

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= \Phi_1(x-\mathrm{i}\theta\sigma\bar\theta,\theta) = e^{-\mathrm{i}\theta\sigma\bar\theta\partial}\Phi_1(x,\theta)\,,\\ \bar\Phi(x,\theta,\bar\theta) &= \bar\Phi_2(x+\mathrm{i}\theta\sigma\bar\theta,\bar\theta) = e^{\mathrm{i}\theta\sigma\bar\theta\partial}\bar\Phi_2(x,\bar\theta)\,,\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}\Phi_1(x,\theta) &= A(x) + \theta^\alpha\psi_\alpha(x) + \theta^\alpha\theta_\alpha F(x)\,,\\ \bar\Phi_2(x,\bar\theta) &= \bar A(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x) + \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\bar F(x)\,.\end{aligned} \tag{2.7}$$

The corresponding covariant derivatives and further technical material on the superfield formalism are collected in appendix A. To a chiral classical superfield $\Phi_1(x_1,\theta_1) \equiv \Phi_1(1)$ one defines the Fourier-transform as[1]

$$\Phi_1(1) = \int dp\, e^{\mathrm{i}px_1}\tilde\Phi_1(p,\theta_1)\,, \tag{2.8}$$

where the coefficients of $\tilde\Phi_1(p,\theta_1)$ belong to $\mathcal{S}\,(\mathbb{R}^4)$. The "inverse" reads

$$\begin{aligned}\tilde\Phi_1(p,\theta_1) &= \int dS_2\, e^{-\mathrm{i}px_2}\delta(\theta_{12})\Phi_1(x_2,\theta_2)\\ &= \int dx_2\, e^{-\mathrm{i}px_2}\Phi_1(x_2,\theta_1)\,,\end{aligned} \tag{2.9}$$

where dS_2 and $\delta(\theta_{12})$ are given in appendix A. Corresponding to Filk [22], one associates to the classical chiral superfield $\Phi_1(x_1,\theta_1)$ the following superoperator:

$$\begin{aligned}\Phi_1(q_1,\theta_1) &= \int dS_2 \int dk\; \mathrm{T}(k)e^{-\mathrm{i}kx_2}\delta(\theta_{12})\Phi_1(x_2,\theta_2)\\ &= \int dx_2 \int dk\; \mathrm{T}(k)e^{-\mathrm{i}kx_2}\Phi_1(x_2,\theta_1)\\ &= \int dk\; \mathrm{T}(k)\tilde\Phi_1(k,\theta_1)\,.\end{aligned} \tag{2.10}$$

[1]We use the notation $dx = d^4x$, $dk = d^4k/(2\pi)^4$

The trace operation allows to recover the classical chiral superfield $\Phi_1(x_1, \theta_1)$

$$\Phi_1(1) = \int dk\, e^{\mathrm{i}kx_1}\, \mathrm{tr}\left[\Phi_1(q_1, \theta_1) T^+(k)\right]. \tag{2.11}$$

Following Filk's idea we are now able to construct a $\star$-product, the so-called Moyal product, of two classical superfields

$$(\Phi_1 \star \Phi_1)(1) = \int dk\, e^{\mathrm{i}kx_1}\, \mathrm{tr}\Big[\Phi_1(q_1, \theta_1)\Phi_1(q_1, \theta_1) T^+(k)\Big]. \tag{2.12}$$

With the relations (2.3)–(2.5), eq. (2.12) becomes

$$(\Phi_1 \star \Phi_1)(1) = \int dk_1 \int dk_2 e^{\mathrm{i}(k_1+k_2)x_1} e^{-\mathrm{i}k_1 \times k_2} \tilde{\Phi}_1(k_1, \theta_1)\tilde{\Phi}_1(k_2, \theta_1), \tag{2.13}$$

and with (2.9) there follows:

$$\begin{aligned} (\Phi_1 \star \Phi_1)(1) = & \int dk_1 \int dk_2 e^{\mathrm{i}(k_1+k_2)x_1} e^{-\mathrm{i}k_1 \times k_2} \int dS_{1'} \times \\ & \times \int dS_{2'}\, e^{-\mathrm{i}k_1 x_{1'} - \mathrm{i}k_2 x_{2'}} \delta(\theta_{11'})\delta(\theta_{12'})\Phi_1(x_{1'}, \theta_{1'})\Phi_1(x_{2'}, \theta_{2'}). \end{aligned} \tag{2.14}$$

Additionally, one has also

$$(\Phi_1 \star \Phi_1 \star \Phi_1)(1) = \int dk\, e^{\mathrm{i}kx_1}\, \mathrm{tr}\Big[\Phi_1(q_1, \theta_1)\Phi_1(q_1, \theta_1)\Phi_1(q_1, \theta_1) T^+(k)\Big]. \tag{2.15}$$

Repeating the same steps as before, eq. (2.15) may be rewritten as

$$\begin{aligned} (\Phi_1 \star \Phi_1 \star \Phi_1)(1) = & \int dk_1 \int dk_2 \int dk_3\, e^{\mathrm{i}(k_1+k_2+k_3)x_1} e^{-\mathrm{i}\sum_{i<j}^3 k_i \times k_j} \times \\ & \times \tilde{\Phi}_1(k_1, \theta_1)\tilde{\Phi}_1(k_2, \theta_1)\tilde{\Phi}_1(k_3, \theta_1) \\ = & \int dk_1 \int dk_2 \int dk_3\, e^{\mathrm{i}(k_1+k_2+k_3)x_1} e^{-\mathrm{i}\sum_{i<j}^3 k_i \times k_j} \times \\ & \times \int dS_{1'} \int dS_{2'} \int dS_{3'}\, e^{-\mathrm{i}k_1 x_{1'} - \mathrm{i}k_2 x_{2'} - \mathrm{i}k_3 x_{3'}} \delta(\theta_{11'})\delta(\theta_{12'})\delta(\theta_{13'}) \times \\ & \times \Phi_1(x_{1'}, \theta_{1'})\Phi_1(x_{2'}, \theta_{2'})\Phi_1(x_{3'}, \theta_{3'}). \end{aligned} \tag{2.16}$$

The $\Phi_1^{\star 3}$ is required to describe the corresponding interactions of the WZ-model. With the functional derivative for chiral superfields

$$\frac{\delta \Phi_1(1)}{\delta \Phi_1(2)} = \delta_S(1,2) = \delta(\theta_{12})\delta(x_1 - x_2) = -\frac{1}{4}\theta_{12}^2 \delta(x_1 - x_2), \tag{2.17}$$

one checks that the above definitions imply also

$$\frac{\delta}{\delta \Phi_1(2)} \int dS_1\, (\Phi_1 \star \Phi_1 \star \Phi_1)(1) = 3(\Phi_1 \star \Phi_1)(2). \tag{2.18}$$

Finally, from (2.13) we get the useful relation

$$\int dS_1\, (\Phi_1 \star \Phi_1)(1) = \int dS_1\, \Phi_1(1)\Phi_1(1). \tag{2.19}$$

JHEP10(2000)046

3. Superfield formulation of the non-commutative Wess-Zumino model at the tree level

In four-dimensional Minkowskian space-time the Wess-Zumino model in terms of superfields is defined at the tree level by the following action [53]

$$\begin{aligned}\Gamma^{(0)} &= \Gamma^{(0)}_{\text{kin}} + \Gamma^{(0)}_{m} + \Gamma^{(0)}_{I} \\ &= \frac{1}{16}\int dV\, \bar{\Phi}\Phi + \frac{m}{8}\left[\int dS\, \Phi^2 + \int d\bar{S}\, \bar{\Phi}^2\right] + \frac{g}{48}\left[\int dS\, \Phi^{\star 3} + \int d\bar{S}\, \bar{\Phi}^{\star 3}\right].\end{aligned} \tag{3.1}$$

Since Φ and $\bar{\Phi}$ are chiral and anti-chiral superfields the kinetic part of $\Gamma^{(0)}$ can be rewritten as

$$\Gamma^{(0)}_{\text{kin}} = \frac{1}{16}\int dS\, \bar{D}^2\bar{\Phi}(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta}) = \frac{1}{16}\int d\bar{S}\, \bar{\Phi}(x,\theta,\bar{\theta})D^2\Phi(x,\theta,\bar{\theta})\,. \tag{3.2}$$

Carrying out the θ and $\bar{\theta}$ integration, which is in fact a differentiation, one gets always the last component of a superfield (products of superfields are again superfields) which furnishes the component formulation of supersymmetric field models. In order to derive the corresponding superfield propagators one uses the Legendre transformation between the functional for connected Green functions Z_c and the vertex functional $\Gamma^{(0)}$. Introducing external chiral and anti-chiral sources J and $\bar{J}$ we have

$$Z_c[J,\bar{J}] = \Gamma^{(0)}[\Phi,\bar{\Phi}] + \int dS J\Phi + \int d\bar{S}\bar{J}\bar{\Phi}\,, \tag{3.3}$$

with

$$\frac{\delta\Gamma^{(0)}}{\delta\Phi} = -J\,, \qquad \frac{\delta\Gamma^{(0)}}{\delta\bar{\Phi}} = -\bar{J}\,, \tag{3.4}$$

and

$$\frac{\delta Z_c}{\delta J} = \Phi\,, \qquad \frac{\delta Z_c}{\delta \bar{J}} = \bar{\Phi}\,. \tag{3.5}$$

Solving eq. (3.4) for $\Phi = \Phi[J,\bar{J}]$ and $\bar{\Phi} = \bar{\Phi}[J,\bar{J}]$, and using the bilinear part of the action, one gets the desired superfield propagators

$$\begin{aligned}\langle T\Phi(1)\Phi(2)\rangle_{(0)} &= \frac{\delta}{\mathrm{i}\delta J(1)}\Phi[J,\bar{J}](2) \\ &= \frac{4\mathrm{i}m\delta_S(1,2)}{\Box + m^2}\,,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\left\langle T\Phi(1)\bar{\Phi}(2)\right\rangle_{(0)} &= \frac{\delta}{\mathrm{i}\delta J(1)}\bar{\Phi}[J,\bar{J}](2) \\ &= -\frac{\mathrm{i}D_2^2\delta_S(1,2)}{\Box + m^2} \\ &= -\frac{\mathrm{i}\bar{D}_1^2\delta_{\bar{S}}(1,2)}{\Box + m^2}\,,\end{aligned} \tag{3.7}$$

and

$$\left\langle T\bar{\Phi}(1)\bar{\Phi}(2)\right\rangle_{(0)} = \frac{4\mathrm{i}m\delta_{\bar{S}}(1,2)}{\Box + m^2}\,. \tag{3.8}$$

Having defined the WZ-model at the tree level with its superfield propagators we are now able to discuss radiative corrections at the one loop level with the help of the Gell-Mann Low formula in terms of superfields.

4. One loop self-energy corrections

The one loop calculations for the self-energy are governed by the Gell-Mann Low formula [53]

$$G(1,\ldots,n) = \langle T\Phi(1)\cdots\Phi(n)\rangle = R\frac{\left\langle T\Phi(1)\cdots\Phi(n)e^{\mathrm{i}\Gamma_I}\right\rangle_{(0)}}{\left\langle Te^{\mathrm{i}\Gamma_I}\right\rangle_{(0)}}\,, \tag{4.1}$$

where we use for Γ_I the "deformed" interaction of the form

$$\Gamma_I = \frac{g}{48}\left[\int dS\,\Phi\star\Phi\star\Phi + \int d\bar{S}\,\bar{\Phi}\star\bar{\Phi}\star\bar{\Phi}\right]. \tag{4.2}$$

No attempt is made to prove formula (4.1). In our approach we define the model intuitively by (4.1) with a deformed interaction. In performing "modified" Wick contractions we will see that our procedure gives a meaningful result which is in agreement with a recent analysis in components [51].

The main advantage of our superfield procedure is the fact that one gets a very compact result in form of supergraphs. In order to demonstrate the power of the superfield formulation it is sufficient to discuss one representative. For this reason we calculate the one loop graph with one chiral and one anti-chiral external leg. Up to some numerical factors the corresponding contribution is given by

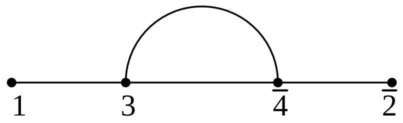


Figure 1: Self-energy graph

$$G(1,\bar{2}) \sim \left\langle T\Phi(1)\bar{\Phi}(2)\left[\int dS_3\,\Phi^{\star 3}(3) + \int d\bar{S}_3\,\bar{\Phi}^{\star 3}(3)\right]\times\right.$$
$$\left.\times\left[\int dS_4\,\Phi^{\star 3}(4) + \int d\bar{S}_4\,\bar{\Phi}^{\star 3}(4)\right]\right\rangle_{(0)}\,. \tag{4.3}$$

By Wick contractions in the presence of deformed interactions we calculate the graph shown in figure 1, which is one of the four contributions corresponding to[2] (4.3).

[2]For further details in the commutative case see [53].

We find

$$
\begin{aligned}
G(1,\bar{2})_{3\bar{4}} \sim & \int dS_3 \int \prod_{i=1}^{3} dk_i \, e^{\mathrm{i}(k_1+k_2+k_3)x_3} e^{-\mathrm{i}\sum_{i<j} k_i \times k_j} \int dx_{31} \int dx_{32} \times \\
& \times \int dx_{33} \, e^{-\mathrm{i}k_1 x_{31} - \mathrm{i}k_2 x_{32} - \mathrm{i}k_3 x_{33}} \int d\bar{S}_4 \int \prod_{i=1}^{3} dk'_i \, e^{\mathrm{i}\left(k'_1+k'_2+k'_3\right)x_4} e^{-\mathrm{i}\sum_{i<j} k'_i \times k'_j} \times \\
& \times \int dx_{41} \int dx_{42} \int dx_{43} \, e^{-\mathrm{i}k'_1 x_{41} - \mathrm{i}k'_2 x_{42} - \mathrm{i}k'_3 x_{43}} \left\langle T\Phi(1)\Phi(3)\right\rangle_{(0)} \times \\
& \times \left(\left\langle T\Phi(3)\bar{\Phi}(4)\right\rangle_{(0)}\right)^2 \left\langle T\bar{\Phi}(4)\bar{\Phi}(2)\right\rangle_{(0)} .
\end{aligned} \tag{4.4}
$$

The required free superfield propagators are defined in eqs. (3.6)–(3.8). A straightforward but lengthy calculation leads to[3]

$$
G(1,\bar{2})_{3\bar{4}} \sim \int dp_1 \, e^{\mathrm{i}p_1(x_1-x_2)} \frac{1}{\left(p_1^2 - m^2\right)^2} \Gamma_{\Phi\bar{\Phi}}(p_1) \,, \tag{4.5}
$$

where $\Gamma_{\Phi\bar{\Phi}}$ is the self-energy 1PI-vertex, in the one loop approximation (up to some numerical factors) given by

$$
\begin{aligned}
\Gamma^{(1)}_{\Phi\bar{\Phi}}(p_1) & \sim \int dk \, \frac{1}{2} \left[1 + \cos(2p_1 \times k)\right] \frac{D_2^2(p_1-k)\tilde{\delta}_S(1,2)}{(p_1-k)^2 - m^2} \frac{D_2^2(k)\tilde{\delta}_S(1,2)}{k^2 - m^2} \\
& = \int dk \, \cos^2(p_1 \times k) \, \frac{D_2^2(p_1-k)\tilde{\delta}_S(1,2)}{(p_1-k)^2 - m^2} \frac{D_2^2(k)\tilde{\delta}_S(1,2)}{k^2 - m^2} \,.
\end{aligned} \tag{4.6}
$$

Using [53]

$$
D_2^2(p)\tilde{\delta}_S(1,2) = \bar{D}_1^2(p)\tilde{\delta}_{\bar{S}}(1,2) = e^{E_{12}p} \,, \tag{4.7}
$$

one gets finally

$$
\Gamma^{(1)}_{\Phi\bar{\Phi}}(p_1) \sim e^{E_{12}p_1} \int dk \, \frac{1}{(p_1-k)^2 - m^2} \frac{1}{k^2 - m^2} \cos^2(p_1 \times k) \,, \tag{4.8}
$$

where E_{12} is defined by

$$
E_{12} = \theta_1 \sigma \bar{\theta}_1 + \theta_2 \sigma \bar{\theta}_2 - 2\theta_1 \sigma \bar{\theta}_2 \,. \tag{4.9}
$$

The result in the form of eq. (4.8) shows in a very elegant manner that the total "undeformed" θ-dependence is encoded in the exponent E_{12}, whereas the remaining Feynman integral represents the "component" result.

From (4.6) it is seen that the one loop self-energy corrections are just the sum of a usual planar contribution (the "1" term in $[\cdots]$ of (4.6)) and a non-planar contribution (the $\cos(\cdot)$ term in $[\cdots]$). The planar contribution contains the expected logarithmically divergent wave function renormalization [53, 51]. Finally we must show that the non-planar integral

$$
I(p,\tilde{p}) = \int dk \, \frac{e^{\mathrm{i}k\tilde{p}}}{\left((p-k)^2 - m^2 + \mathrm{i}\epsilon\right)\left(k^2 - m^2 + \mathrm{i}\epsilon\right)} \,, \tag{4.10}
$$

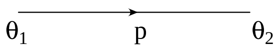
[3] We omit the $\mathrm{i}\epsilon$-prescription.

leads to a finite result for non-vanishing $\tilde{p}^\mu = \Sigma^{\mu\nu} p_\nu$. The calculations are given in appendix B. We find

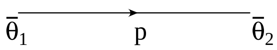
$$I(p,\tilde{p}) = -\frac{2\mathrm{i}}{(4\pi)^2} \int_0^1 dx\, K_0\Big(\sqrt{(m^2 - p^2 x(1-x))(-\tilde{p}^2)}\Big)\,. \tag{4.11}$$

5. Feynman rules in momentum space and one loop vertex corrections

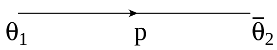
In order to be complete this section is devoted to discuss the one loop vertex corrections directly with the help of the Feynman rules in momentum space. With the conventions of [53] one has the following Feynman rules in momentum space



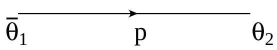
$$\langle T\Phi(1)\Phi(2)\rangle\tilde{}_{(0)} = \frac{4\mathrm{i}m\tilde{\delta}_S(1,2)}{p^2 - m^2 + \mathrm{i}\epsilon}\,, \tag{5.1}$$



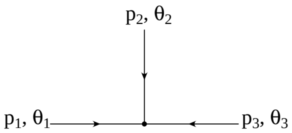
$$\langle T\bar{\Phi}(1)\bar{\Phi}(2)\rangle\tilde{}_{(0)} = \frac{4\mathrm{i}m\tilde{\delta}_{\bar{S}}(1,2)}{p^2 - m^2 + \mathrm{i}\epsilon}\,,$$



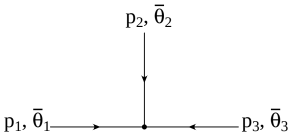
$$\langle T\Phi(1)\bar{\Phi}(2)\rangle\tilde{}_{(0)} = \frac{\mathrm{i}D_2^2(p)\tilde{\delta}_S(1,2)}{p^2 - m^2 + \mathrm{i}\epsilon} = \frac{\mathrm{i}\bar{D}_1^2(p)\tilde{\delta}_{\bar{S}}(1,2)}{p^2 - m^2 + \mathrm{i}\epsilon}\,, \tag{5.2}$$



$$\langle T\bar{\Phi}(1)\Phi(2)\rangle\tilde{}_{(0)} = \frac{\mathrm{i}\bar{D}_2^2(p)\tilde{\delta}_{\bar{S}}(1,2)}{p^2 - m^2 + \mathrm{i}\epsilon} = \frac{\mathrm{i}D_1^2(p)\tilde{\delta}_S(1,2)}{p^2 - m^2 + \mathrm{i}\epsilon}\,, \tag{5.3}$$



$$\frac{1}{8} g (2\pi)^4 \delta\left(p_1 + p_2 + p_3\right) \delta\left(\theta_{12}\right) \delta\left(\theta_{13}\right) \cos\left(p_2 \times p_3\right), \tag{5.4}$$



$$\frac{1}{8} g (2\pi)^4 \delta\left(p_1 + p_2 + p_3\right) \delta\left(\bar{\theta}_{12}\right) \delta\left(\bar{\theta}_{13}\right) \cos\left(p_2 \times p_3\right). \tag{5.5}$$

With the Feynman rules (5.1)–(5.5) one easily confirms the result of eq. (4.6). Additionally, one can state the non-renormalization theorem for $\Gamma_{\Phi\Phi}$ and $\Gamma_{\Phi\Phi\Phi}$ with only chiral (or anti-chiral) internal lines. A possible self-energy correction of this type is

$$\Gamma_{\Phi\Phi} \sim g^2 \int dk\, \frac{4\mathrm{i}m\tilde{\delta}_S(1,2)}{(p+k)^2 - m^2 + \mathrm{i}\epsilon} \frac{4\mathrm{i}m\tilde{\delta}_S(2,1)}{k^2 - m^2 + \mathrm{i}\epsilon} \cos^2\left(p \times k\right). \tag{5.6}$$

Due to $\tilde{\delta}_S(1,2)\tilde{\delta}_S(2,1) \sim \frac{1}{16}(\theta_{12}^2)^2 = 0$ one has: $\Gamma_{\Phi\Phi} = 0$. In a similar manner one can show that also the one loop vertex correction shown in figure $2a$ vanishes

$$\begin{aligned}\Gamma_{\Phi\Phi\Phi} \sim g^3 \int dk\, (4\mathrm{i}m)^3 \frac{\tilde{\delta}_S(1,2)}{(p_1+k)^2 - m^2 + \mathrm{i}\epsilon} \frac{\tilde{\delta}_S(2,3)}{(p_1+p_2+k)^2 - m^2 + \mathrm{i}\epsilon} \frac{\tilde{\delta}_S(3,1)}{k^2 - m^2 + \mathrm{i}\epsilon} \times \\ \times \cos\left(p_1 \times k\right) \cos\left(p_2 \times p_1 + p_2 \times k\right) \cos\left(p_1 \times k + p_2 \times k\right). \end{aligned} \tag{5.7}$$

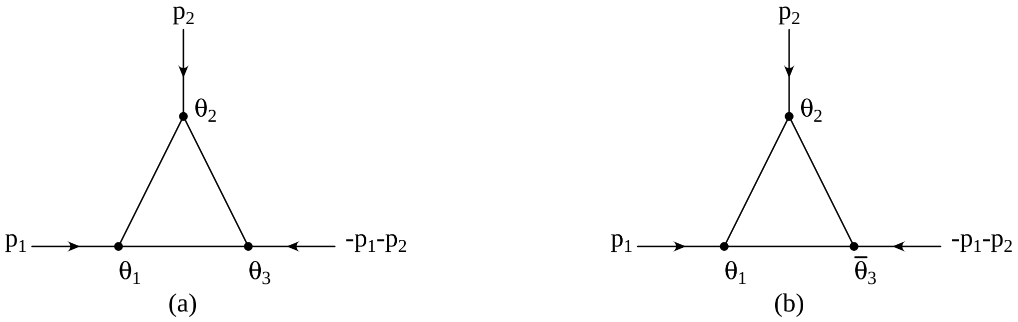


Figure 2: One loop 3-point graphs

Using the appendix A we find that the product of the three $\tilde{\delta}$-functions vanishes. However, there could exist a non-vanishing one loop correction if one allows mixed propagators, see figure 2b. Applying the above Feynman rules one gets

$$\Gamma_{\Phi\Phi\bar{\Phi}} \sim g^3 \int dk \, \frac{4\mathrm{i}m\tilde{\delta}_S(1,2)}{(p_1+k)^2-m^2+\mathrm{i}\epsilon} \frac{\mathrm{i}D_3^2(p_1+p_2+k)\tilde{\delta}_S(2,3)}{(p_1+p_2+k)^2-m^2+\mathrm{i}\epsilon} \frac{\mathrm{i}D_3^2(k)\tilde{\delta}_S(3,1)}{k^2-m^2+\mathrm{i}\epsilon} \times$$
$$\times \frac{1}{4}\Big[\cos\left(p_1\times p_2\right)+\cos\left(p_2\times p_1+2(p_1+p_2)\times k\right)+$$
$$+\cos\left(p_2\times p_1-2p_1\times k\right)+\cos\left(p_2\times p_1+2p_2\times k\right)\Big], \qquad (5.8)$$

where we have separated the vertex correction in a planar (the $\sim \cos(p_1\times p_2)$ term, which does not depend on the internal momentum) and a non-planar contribution. Using (A.10) we conclude that the integral (5.8) is finite, since it goes asymptotically like $\sim 1/k^6$. This is required in order to secure stability of the classical action.

6. Renormalization at all orders and conclusion

Using the same integration techniques as in appendix B one can prove that the superficial integration encoded in any non-planar 1PI Feynman (sub)graph γ produces a Bessel function $K_{.}(.)$ depending on the external momenta of γ. The Bessel function $K_{.}(.)$ tends exponentially to zero if the external momenta of γ become large. This suffices to render the integration of a graph γ' containing γ as a subgraph UV finite. The only problem could be an IR singularity of the Bessel function. However, since in the commutative Wess-Zumino model there are only logarithmic divergences (in the 2-point function), and because the only difference on the non-commutative space are phase factors, there can only be a logarithmic IR singularity coming from the Bessel function. Nested logarithmic singularities are IR-integrable, as it was explicitly demonstrated in [30]. In conclusion, a graph in the WZ model which contains non-planar sectors leads always to a convergent integral. In particular, a non-planar graph in the standard sense of the commutative world is always convergent. Divergences come only from completely planar graphs, and they are subtracted e.g.

by the BPHZ procedure as in [53]. We therefore conclude that the Wess-Zumino model on non-commutative Minkowski space is renormalizable to all loop orders, a result which was already obtained in [51] and conjectured in [29]. Note that the β functions of the non-commutative and the commutative theory differ because the standard non-planar graphs become finite on the non-commutative space.

In this paper we have demonstrated the strength of the superfield formalism. Especially we would like to emphasize that the superfield formalism enables us to apply eq. (4.7), which lowers the degree of divergence (both IR and UV) by two. Furthermore is the number of graphs to be computed considerable lower than in the work of [51]. We believe that this formalism will prove useful for further investigations, in particular for super Yang-Mills theories on non-commutative $\mathbb{R}^4$.

A. Conventions and useful formulae in superspace

Let us briefly summarize some of the conventions and rules concerning supersymmetry and superspace (most of the rules are taken from [53]).

Metrics, index transport and scalar products. The metric tensor of Minkowski space is given by $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and we use the following spinor metric:

$$\begin{aligned} \epsilon^{\alpha\beta} &= \mathrm{i}\sigma^2 = \epsilon^{\dot\alpha\dot\beta}\,, \\ \epsilon_{\alpha\beta} &= -\mathrm{i}\sigma^2 = \epsilon_{\dot\alpha\dot\beta}\,, \\ \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} &= \delta_\alpha^\gamma\,, \\ \theta\eta &= \theta^\alpha\eta_\alpha\,, \qquad \bar\theta\bar\eta = \bar\theta_{\dot\alpha}\bar\eta^{\dot\alpha}\,, \\ \theta_\alpha &= \epsilon_{\alpha\beta}\theta^\beta\,, \qquad \bar\theta^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}\,. \end{aligned} \tag{A.1}$$

Pauli matrices

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2 &= \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} \vec\sigma &= \left(\sigma^1, \sigma^2, \sigma^3\right), \\ \sigma^\mu &= (1\!\!\perp_2, \vec\sigma) = (\sigma^\mu)_{\alpha\dot\beta}\,, \\ \bar\sigma^\mu &= (1\!\!\perp_2, -\vec\sigma) = (\bar\sigma^\mu)^{\dot\alpha\beta}\,. \end{aligned} \tag{A.3}$$

Covariant derivatives

$$\begin{aligned} D_\alpha &= \partial_\alpha - \mathrm{i}(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu\,, \qquad \bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} + \mathrm{i}\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu\,, \\ \{D_\alpha, \bar D_{\dot\beta}\} &= 2\mathrm{i}(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu\,, \qquad \{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0\,. \end{aligned} \tag{A.4}$$

Integration

$$\int dV\,\Phi = \int d^4x D^2\bar{D}^2\,\Phi \qquad \text{for any superfield } \Phi\,,$$
$$\int dS\,\Phi = \int d^4x D^2\,\Phi \qquad \text{for a chiral superfield } \Phi \text{ (i.e. } \bar{D}\Phi = 0)\,,$$
$$\int d\bar{S}\,\bar{\Phi} = \int d^4x \bar{D}^2\,\bar{\Phi} \qquad \text{for an anti-chiral superfield } \bar{\Phi} \text{ (i.e. } D\bar{\Phi} = 0)\,. \tag{A.5}$$

Functional differentiation, delta-functions and representations

$$\frac{\delta\Phi(1)}{\delta\Phi(2)} = \delta_V(1,2) \qquad \text{for any superfield}\,,$$
$$\frac{\delta\Phi_1(1)}{\delta\Phi_1(2)} = \delta_S(1,2) \qquad \text{for a chiral superfield}\,,$$
$$\frac{\delta\bar{\Phi}_2(1)}{\delta\bar{\Phi}_2(2)} = \delta_{\bar{S}}(1,2) \qquad \text{for an anti-chiral superfield}\,, \tag{A.6}$$

the numbers denoting points in superspace (for instance, (1) is a shorthand-notation for $((x^\mu)_1, (\theta_\alpha)_1, (\bar{\theta}^{\dot{\alpha}})_1)$). The above delta-functions are in position space given by:

$$\delta_V(1,2) = \frac{1}{16}\theta_{12}^2\bar{\theta}_{12}^2\delta^4(x_1 - x_2)\,, \qquad \int dV_1\,\Phi(1)\delta_V(1,2) = \Phi(2)\,, \tag{A.7}$$
$$\delta_S(1,2) = -\frac{1}{4}\theta_{12}^2\delta^4(x_1 - x_2)\,, \qquad \int dS_1\,\Phi(1)\delta_S(1,2) = \Phi(2)\,, \tag{A.8}$$
$$\delta_{\bar{S}}(1,2) = -\frac{1}{4}\bar{\theta}_{12}^2\delta^4(x_1 - x_2)\,, \qquad \int d\bar{S}_1\,\bar{\Phi}(1)\delta_{\bar{S}}(1,2) = \bar{\Phi}(2)\,. \tag{A.9}$$

Note that the first equalities of eqs. (A.8) and (A.9) are only valid in the chiral and anti-chiral representation, respectively. The "Fourier-transforms" of the δ-functions in the real representation are

$$\tilde{\delta}_S(1,2) = -\frac{1}{4}\theta_{12}^2 e^{-(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1)p}\,, \qquad D_1^2(p)\tilde{\delta}_S(1,2) = e^{-E_{12}p}\,, \tag{A.10}$$
$$\tilde{\delta}_{\bar{S}}(1,2) = -\frac{1}{4}\bar{\theta}_{12}^2 e^{-(\theta_1\sigma\bar{\theta}_2 - \theta_2\sigma\bar{\theta}_1)p}\,, \qquad \bar{D}_1^2(p)\tilde{\delta}_{\bar{S}}(1,2) = e^{E_{12}p}\,, \tag{A.11}$$

where

$$E_{12} = \theta_1\sigma\bar{\theta}_1 + \theta_2\sigma\bar{\theta}_2 - 2\theta_1\sigma\bar{\theta}_2\,,$$
$$\theta_{12}^2 = (\theta_1 - \theta_2)^2\,, \qquad \bar{\theta}_{12}^2 = (\bar{\theta}_1 - \bar{\theta}_2)^2\,. \tag{A.12}$$

Different representations (real and (anti)chiral) of (anti)chiral superfields are connected by the following relations:

$$\Phi(x,\theta,\bar{\theta}) = \Phi_1(x - \mathrm{i}\theta\sigma\bar{\theta}, \theta) = e^{-\mathrm{i}\theta\sigma\bar{\theta}\partial}\Phi_1(x,\theta)\,,$$
$$\bar{\Phi}(x,\theta,\bar{\theta}) = \bar{\Phi}_2(x + \mathrm{i}\theta\sigma\bar{\theta}, \bar{\theta}) = e^{\mathrm{i}\theta\sigma\bar{\theta}\partial}\bar{\Phi}_2(x,\bar{\theta})\,, \tag{A.13}$$

JHEP10(2000)046

1 denoting the chiral and 2 the anti-chiral representation; also, (anti)chiral fields have a simplified θ-expansion:

$$\begin{aligned} \Phi_1(x,\theta) &= A(x) + \theta^\alpha \psi_\alpha(x) + \theta^\alpha \theta_\alpha F(x)\,, \\ \bar{\Phi}_2(x,\bar{\theta}) &= \bar{A}(x) + \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}(x) + \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \bar{F}(x)\,, \end{aligned} \tag{A.14}$$

and the covariant derivatives are given by:

$$\begin{aligned} (D_\alpha \Phi)_1 &= \big(\partial_\alpha - 2\mathrm{i}(\sigma^\mu \bar{\theta})_\alpha \partial_\mu\big)\,\Phi_1\,, \qquad (\bar{D}_{\dot{\alpha}}\Phi)_1 = -\bar{\partial}_{\dot{\alpha}} \Phi_1\,, \\ (D_\alpha \Phi)_2 &= \partial_\alpha \Phi_2\,, \qquad (\bar{D}_{\dot{\alpha}}\Phi)_2 = \big(-\bar{\partial}_{\dot{\alpha}} + 2\mathrm{i}(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\big)\,\Phi_2\,. \end{aligned} \tag{A.15}$$

B. Calculation of the non-planar self-energy graph integral

We are going to compute

$$I(p,\tilde{p}) = \lim_{\epsilon\to 0} \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{\mathrm{i}(\tilde{p}_0 k_0 - \vec{\tilde{p}}\cdot\vec{k})}}{((k_0-p_0)^2 - |\vec{k}-\vec{p}|^2 - m^2 + \mathrm{i}\epsilon)(k_0^2 - |\vec{k}|^2 - m^2 + \mathrm{i}\epsilon)}\,.$$

We apply Zimmermann's ϵ trick [54]:

$$\begin{aligned} I(p,\tilde{p}) &= \lim_{\epsilon\to 0} \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{\mathrm{i}(\tilde{p}_0 k_0 - \vec{\tilde{p}}\cdot\vec{k})}}{((k_0-p_0)^2 - (|\vec{k}-\vec{p}|^2+m^2)(1-\mathrm{i}\epsilon))(k_0^2 - (|\vec{k}|^2+m^2)(1-\mathrm{i}\epsilon))} \\ &= \lim_{\epsilon\to 0} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx\, \frac{e^{\mathrm{i}(\tilde{p}_0 k_0 - \vec{\tilde{p}}\cdot\vec{k})}}{((k_0^2 - 2p_0k_0x + p_0^2x) + (|\vec{k}|^2 - 2\vec{p}\cdot\vec{k}x + |\vec{p}|^2x + m^2)(\mathrm{i}\epsilon - 1))^2} \\ &= \lim_{\epsilon\to 0} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \int_0^\infty d\alpha\; \alpha(\epsilon'-\mathrm{i})^2 \times \\ &\qquad \times e^{-\alpha(\epsilon'-\mathrm{i})((k_0^2 - 2p_0k_0x + p_0^2x) + (|\vec{k}|^2 - 2\vec{p}\cdot\vec{k}x + |\vec{p}|^2x + m^2)(\mathrm{i}\epsilon-1)) + \mathrm{i}(\tilde{p}_0k_0 - \vec{\tilde{p}}\cdot\vec{k})} \\ &= \lim_{\epsilon\to 0} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \int_0^\infty d\alpha\; \alpha(\epsilon'-\mathrm{i})^2\, e^{\mathrm{i}(\tilde{p}_0p_0 - \vec{\tilde{p}}\cdot\vec{p})x} \times \\ &\qquad \times e^{-\alpha(\epsilon'-\mathrm{i})(k_0 - p_0x - \frac{\mathrm{i}\tilde{p}_0}{2\alpha(\epsilon'-\mathrm{i})})^2 - \alpha(\epsilon-\epsilon'+\mathrm{i}+\epsilon\epsilon'\mathrm{i})|\vec{k} - \vec{p}x + \frac{\mathrm{i}\vec{\tilde{p}}}{2\alpha(\epsilon-\epsilon'+\mathrm{i}+\epsilon\epsilon'\mathrm{i})}|^2} \times \\ &\qquad \times e^{-\alpha(\epsilon'-\mathrm{i})p_0^2x(1-x) - \alpha(\epsilon-\epsilon'+\mathrm{i}+\epsilon\epsilon'\mathrm{i})(|\vec{p}|^2x(1-x)+m^2) - \frac{\tilde{p}_0^2}{4\alpha(\epsilon'-\mathrm{i})} - \frac{|\vec{\tilde{p}}|^2}{4\alpha(\epsilon-\epsilon'+\mathrm{i}+\epsilon\epsilon'\mathrm{i})}}\,. \end{aligned}$$

For $\epsilon' < \epsilon$ we perform the gaussian k integration:

$$\begin{aligned} I(p,\tilde{p}) &= \lim_{\epsilon\to 0} \frac{1}{(4\pi)^2} \int_0^1 dx \int_0^\infty \frac{d\alpha}{\alpha} \Big(\frac{\epsilon'-\mathrm{i}}{\epsilon-\epsilon'+\mathrm{i}+\epsilon\epsilon'\mathrm{i}}\Big)^{3/2} \mathrm{e}^{\mathrm{i}(\tilde{p}_0p_0 - \vec{\tilde{p}}\cdot\vec{p})x} \times \\ &\qquad \times e^{-\alpha(\epsilon'-\mathrm{i})p_0^2x(1-x) - \alpha(\epsilon-\epsilon'+\mathrm{i}+\epsilon\epsilon'\mathrm{i})(|\vec{p}|^2x(1-x)+m^2) - \frac{\tilde{p}_0^2}{4\alpha(\epsilon'-\mathrm{i})} - \frac{|\vec{\tilde{p}}|^2}{4\alpha(\epsilon-\epsilon'+\mathrm{i}+\epsilon\epsilon'\mathrm{i})}}\,. \end{aligned}$$

The α integration yields a result independent of $\epsilon' < \epsilon$:

$$\begin{aligned} I(p,\tilde{p}) &= \lim_{\epsilon\to 0} \frac{2}{(4\pi)^2} \int_0^1 dx \Big(\frac{-1-\mathrm{i}\epsilon}{1+\epsilon^2}\Big)^{3/2} e^{\mathrm{i}(\tilde{p}_0p_0 - \vec{\tilde{p}}\cdot\vec{p})x} \\ &\qquad \times K_0\left(\sqrt{(p_0^2x(1-x) - (1-\mathrm{i}\epsilon)(|\vec{p}|^2x(1-x) + m^2))\left(\tilde{p}_0^2 - \frac{1+\mathrm{i}\epsilon}{1+\epsilon^2}|\vec{\tilde{p}}|^2\right)}\right). \end{aligned}$$

Now we can put $\epsilon \to 0$ and switch to Minkowskian scalar products:

$$I(p,\tilde{p}) = -\frac{2\mathrm{i}}{(4\pi)^2}\int_0^1 dx\, K_0\Big(\sqrt{(m^2 - p^2x(1-x))(-\tilde{p}^2)}\Big)\,. \tag{B.1}$$

On the mass shell we have $p^2 = m^2$ and $\tilde{p}^2 \le 0$: if the particle moves for instance in 3-direction then $\tilde{p}^2 = -\Sigma_{03}^2 m^2 - \sum_{i=1,2}(\Sigma_{i3}|\vec{p}| + \Sigma_{i0}\sqrt{|\vec{p}|^2 + m^2})^2$.

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